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Radial Kernels Collocation Method for the Solution of Volterra Integro-Differential Equations

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Abstract

Radial kernel interpolation is an advanced method in approximation theory for the construction of higher order accurate interpolants for scattered data up to higher dimensional spaces. In this manuscript, we formulate a radial kernel collocation approach for solving problems involving the Volterra integro-differential equations using two radial kernels: The Generalized Multi-quadrics and the linear Laguerre-Gaussians. This was achieved by simplifying the Volterra integral problem's solution to an algebraic system of equations. The impact of the shape parameter present in every kernel on the method's accuracy is examined and found to be significant. Two examples were used to illustrate the process; the numerical results are shown as tables and graphs. MATLAB 2018a was employed in the process.

Keywords: *Radial Kernel, Volterra Integro-Differential Equations, Generalized Multi-quadrics, Linear Laguerre Gaussians, Collocation Method, System Matrix.*

1. Introduction

Integro-differential equations (IDEs) have emerged as a powerful tool for modeling various phenomena in diverse fields, including physics, engineering, biology, and economics such as population dynamics and chemical reactions, describing wave propagation and heat transfer, analyzing electrical circuits and mechanical systems, designing control systems and signal processing algorithms, modeling epidemic spread and pharmacokinetics, modeling economic growth and financial markets [1], and so on. These classes of differential equations have proven to be a versatile tool for modeling complex phenomena across various disciplines. Their applications continue to expand, driven by advances in computational power and numerical methods. This demonstrates the significance of IDEs in understanding and analyzing real-world problems, highlighting their potential for future research and applications [2], [3].

A Galerkin meshless approach based on MLS was applied to solve logarithmic and weakly singular boundary integral equations. To solve particular integro-differential models, the authors of [4] have looked at two different RBF collocation techniques; a boundary-type RBF collocation technique and a domain-type RBF collocation method. The discrete Galerkin technique approximates the solution by using the MLS scheme built on dispersed nodal points [5].

In a variable-step size implementation, collocation methods are particularly useful because they yield an approximation of the equation's solution over the entire integration interval. In fact, by evaluating the collocation polynomial, one can easily recover past values that have been lost when step size is changed. Other desirable properties of collocation methods include their high order of convergence, strong stability properties, and flexibility [6]. In fact, if some information about the exact solution's behavior is known, one can choose the collocation functions to better follow that behavior, producing mixed collocation methods, for instance, in the case of VIEs, see [7], and in the case of ordinary differential equations (ODEs), see [8]. It's also important to note that collocation

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methods have significant theoretical implications. For example, many numerical methods are challenging to analyze as discrete schemes, but when recast as collocation-based methods, their analysis can be done elegantly and reasonably simpler. And one notable disadvantage of one-step collocation techniques is that, when they are applied to stiff issues, they experience the order reduction phenomena [9].

The degree of convergence is not constant. For example, when using Runge-Kutta methods based on s-stage collocation on Gauss-Legendre collocation points, the order in the grid points is p = 2s, but for stiff problems, it degenerates to p = s, given that the order is currently internal. Two-step collocation techniques, which have a high uniform order on the whole integration interval, satisfactorily address this issue [10]. Collocation methods are typically more expensive than other groups of procedures when it comes to computational cost. In actuality, the solution of a nonlinear system of dimension m is necessary for a collocation method with m collocation parameters at each time-step. Multistep collocation techniques, which raise the order of convergence at the same computational cost as one-step ones, can be used to overcome this disadvantage [4], [11].

Applying a collocation approach to an integral equation necessitates computing several integrals; hence, appropriate quadrature procedures are required to finish the discretization, resulting in an extra error. Finally, there is still a need for a trustworthy error estimate for collocation techniques for integral equations. Some progress has been made in this area (see the references in [8], [12]), but a lot more work has to be done. The primary findings of one-step collocation techniques are compiled in monograph [13], which were initially documented in the literature. In an effort to increase the order of convergence of classical one-step collocation methods without incurring additional computational costs at each time step and to simultaneously obtain highly stable methods, we have recently proposed radial kernels collocation methods [14] for solving Fredholm integro-differential equations, in which the approximate solution in a fixed number of previous time steps.

Consequently, in order to compute integrals in the method that is provided, which is based on the composite Gauss-Legendre quadrature formula, we need an appropriate integration rule. The researches method simplifies the Volterra integral problem's solution to an algebraic equation system's solution. This new method is meshless because it doesn't require any domain elements. The method is more adaptable for most classes of Volterra integral equations and does not increase the difficulty for higher dimensional issues because of the straightforward adaptation of the radial kernels' collocation method. We also investigate the correctness and efficiency of the proposed method in some Volterra integral equations by way of implementation using examples [2], [15].

The proposed research introduces a novel computational approach by utilizing radial kernels within the collocation method framework to solve Volterra integro-differential equations (VIDEs). While traditional numerical methods such as finite difference and spectral methods have been employed to tackle VIDEs, their limitations in handling complex and non-smooth solutions have prompted the need for more adaptive techniques. The radial kernels collocation method offers greater potentials in terms of flexibility, accuracy, and efficiency in approximating the solutions of nonlinear and higher-dimensional VIDEs, making it particularly suitable for problems with irregular domains. This study integrates the radial basis functions with collocation methods to overcome the challenges associated with traditional techniques, providing a more robust tool for both theoretical analysis and practical applications in diverse scientific fields.

2. Preliminaries

As per [16], Volterra examined the genetic factors while examining a population expansion model. The investigation produced a series of equations in which the integral and differential operators coexisted in a single equation. If all of the integration limits are constant, the peculiar kind of equations is known as an integro-differential equation, or more specifically, a Volterra integro-differential equation [17] and [18]. The following is a k^{th} order linear Volterra integro-differential equation:

$$\sum_{n=0}^{k} P_n(x)u^{(n)}(x) = f(x) + \lambda \int_a^x K(x,t)\,u(t)dt, \qquad (1)$$

where $u^{(n)}(x) = \frac{d^n u}{dx^n}$. It is necessary to define initial conditions $u(0)$, $u'(0)$, …, $u^{(n-1)}(0)$ for the

determination of the particular solution $u(x)$ of the Volterra [19].

The radial kernels used for implementation in this work and their derivatives are defined as follows [19], [20]:

- Generalized Multiquadric:

$\varphi(r) = ((\varepsilon r)^2 + 1)^{5/2},$
$\varphi'(r) = 5\varepsilon^2 r((\varepsilon r)^2 + 1)^{3/2},$
$\varphi''(r) = 15\varepsilon^2((\varepsilon r)^2 + 1)^{\frac{1}{2}}((\varepsilon r)^2 + 1)^3 + 3(\varepsilon r)^2),$

- Linear Laguerre-Gaussians:

$\varphi(r) = e^{-(\varepsilon r)^2}(2 - (\varepsilon r)^2),$
$\frac{\partial}{\partial x}\varphi(r) = 2\varepsilon^2 r e^{-(\varepsilon r)^2}((\varepsilon r)^2 - 3),$
$\frac{\partial^2}{\partial y^2}\varphi(r) = 2\varepsilon^2 e^{-(\varepsilon r)^2}((\varepsilon r)^4 - 9)(\varepsilon r)^2 + 3).$

2.1. Interpolation by Radial Kernels

In many applications, the most used kernels are translational invariant or radial. To be precise, there exist a univariate function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that [21].

$K(x,y) = \varphi(x-y),\ x,y \in \Omega \text{ or } \varphi : \mathbb{R} \geq 0 \to \mathbb{R}$

such that,

$K(x,y) = \varphi(\|x-y\|_2), \qquad x,y \in \Omega$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$. Radial kernels, which also take the name Radial Basis Functions (RBFs), are usually defined with the parameter $\varepsilon > 0$, which is called the shape parameter and is employed to control the scale of the kernel [21], i.e.,

$K(x,y) = \varphi(\varepsilon\|x-y\|_2).$

Rather than using simple distance matrices $B_k(x) = \|x - x_k\|_2$, radial kernels expansion can be used to solve the interpolation problem in $\mathbb{R}^d$ by assuming

$$s_f(x) = \sum_{k=1}^{N} c_k \varphi(\|x - x_k\|_2) \tag{2}$$

The coefficients c_k in (2) are found by enforcing the interpolation conditions, and thus solving the linear system [19],

$$\begin{bmatrix} \varphi(\|x_1 - x_1\|_2) & \varphi(\|x_1 - x_2\|_2) & \cdots & \varphi(\|x_1 - x_N\|_2) \\ \varphi(\|x_2 - x_1\|_2) & \varphi(\|x_2 - x_2\|_2) & \cdots & \varphi(\|x_2 - x_N\|_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(\|x_N - x_1\|_2) & \varphi(\|x_N - x_2\|_2) & \cdots & \varphi(\|x_N - x_N\|_2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix} \tag{3}$$

Note that this system will have a unique solution whenever the system matrix A in (3) above [19].

$$A = \begin{bmatrix} \varphi(\|x_1 - x_1\|_2) & \varphi(\|x_1 - x_2\|_2) & \cdots & \varphi(\|x_1 - x_N\|_2) \\ \varphi(\|x_2 - x_1\|_2) & \varphi(\|x_2 - x_2\|_2) & \cdots & \varphi(\|x_2 - x_N\|_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(\|x_N - x_1\|_2) & \varphi(\|x_N - x_2\|_2) & \cdots & \varphi(\|x_N - x_N\|_2) \end{bmatrix}$$

is non-singular and therefore, the choice of a kernel for which the system matrix is non-singular is also important. A full characterization of the class of all kernels that generate a non-singular system matrix for any set $X = \{x_1, \ldots, x_N\}$ of distinct data sites is not an easy task. We refer readers to [19] for the most commonly used ones. The invertibility conditions of the method could be fully gotten from the same reference above.

We assume that the solution of the k^{th} order Volterra integro-differential equation (1) can be expressed in the form of the radial kernel interpolant

$$u(x) \approx \tilde{u}(x) = \sum_{i=1}^{m} \alpha_i \varphi(\varepsilon\|x - x_i\|_2) \tag{4}$$

$m > k$ and $x \in R$

where φ is a positive definite kernel [6], [14]. The Laguerre Gaussians and Generalized Multi-quadrics kernels will be used for this investigation.

We first of all select $m - k$ collocation points from the m data sites $x_1, \ldots, x_m$. Let $\alpha = \lceil k/2 \rceil$ and $\beta = \lfloor k/2 \rfloor$, and so we use $x_\alpha, \ldots, x_{m-\beta}$ as the $m - k$ collocation points. When we substitute the collocation points into equation (1), we obtain

$$\sum_{n=0}^{k} P_n(x_j) u^{(n)}(x_j) = f(x_j) + \lambda \int_a^{x_j} K(x_j, t)\, u(t)dt, \quad j = \alpha \ldots m - \beta. \tag{5}$$

We now substitute the approximate solution (4) into equation (5) to obtain

$$\sum_{n=0}^{k} P_n(x_j)\left[\sum_{i=1}^{m} c_i \varphi^{(n)}\left(\varepsilon\left\|x_j - x_i\right\|_2\right)\right]$$

$$= f(x_j) + \lambda \int_a^b K(x_j, t) \sum_{i=1}^{m} c_i \varphi(\varepsilon\|t - x_i\|_2)\, dt$$

$j = \alpha, \dots, m - \beta.$

Re-arranging these yields

$$\sum_{i=1}^{m} c_i \left\{\sum_{n=0}^{k} P_n(x_j)\varphi^{(n)}\left(\varepsilon\left\|x_j - x_i\right\|_2\right) - \lambda \int_a^b K(x_j, t)\, \varphi(\varepsilon\|t - x_i\|_2) dt\right\} = f(x_j) \quad (6)$$

The integral in equation (6) is evaluated using a five-point Gauss-Legendre quadrature formula on the interval $[-1, 1]$ as [22] and [23].

$$\int_{-1}^{1} g(x)\, dx = \sum_{i=1}^{5} w_i g(p_i) \quad (7)$$

where the $w_i's$ are the weights and the $p_i's$ are the integration points. Here:

$w_1 = w_5 = 0.23692688,$

$w_2 = w_4 = 0.47862867,$

$w_3 = 0.56888888,$ and

$p_1 = -0.90617984,$

$p_2 = -0.53846931,$

$p_3 = 0.00000000,$

$p_5 = -p_1, p_4 = -p_2.$

To apply the rule over an arbitrary interval $[a, b]$, we use the change of variable

$$t = \frac{a+b}{2} + \frac{a-b}{2}x \text{ and } dt = \frac{a-b}{2}dx.$$

The m coefficients $c_1, \dots, c_m$ of the approximate will require solving a system of m linear equations. Equation (6) gives $m - k$ linear equations in $c_1, \dots, c_m$ while the remaining k equations are obtained by evaluating the approximate solution at the initial conditions [24]. This gives

$$\tilde{u}^{(n)}(x_0) = \sum_{i=1}^{n} c_i \varphi^{(n)}(\varepsilon\|x_0 - x_i\|) = \rho_n \quad (8)$$

$n = 0, \dots, k - 1$

Equations (6) and (8) together yield a system of n equations in m unknowns. The collocation method for the solution of Fredholm integro-differential equation will be implemented using the linear Laguerre-Gaussian and the generalized multi-quadrics [24].

2.2. Smoothness Requirements for the Kernels

According to [25], The MQ and the LLG kernels are infinitely differentiable, meaning they requires the underlying function $u(x)$, which is the solution to the Volterra integro-differential equation, to be infinitely smooth in theory. But in practice, it is enough if the solution $u(x)$ belongs to a Sobolev space $W^{k,p}$, where the solution has continuous partial derivatives up to a certain order k. A smoothness of $k \geq 2$ (i.e., the function has at least two continuous derivatives) is sufficient for convergence.

The shape parameter ε can be tuned to adjust the smoothness requirements, large shape parameter (i.e., $\varepsilon > 1$) results in a flat MQ kernel, making the method better suited for smooth functions, but it can reduce accuracy in regions where the function has steep gradients or irregular behavior. A small shape parameter (i.e., $\varepsilon < 1$) make the kernel more localized and sensitive to irregularities in the solution, allowing the method to handle functions with less smoothness. However, this could lead to ill-conditioning in the system of equations, which needs to be handled carefully [26].

Convergence: The error estimate for the MQ and LLG method typically take the form

$$\|u(x) - u_n(x)\| \leq Ch^{\mu}$$

where $u_n(x)$ is the numerical solution, h is the fill distance (the spacing between collocation points), and μ depends on the smoothness of $u(x)$ and the shape parameter ε. The smoother the solution and the better the choice of ε, the higher the convergence rate μ.

Reducing Smoothness Requirements to Obtain Convergence for both the MQ and LLG kernels while still achieving convergence, the following approaches was explored;

Adaptive Shape Parameter Selection for MQ; choosing a smaller shape parameter can improve the method's ability to approximate fewer smooth functions. Similarly, for LLG, adjusting ε and the degree k of the Laguerre polynomial can allow the method to approximate fewer smooth solutions while maintaining good convergence. Other approaches like Mixed Radial Kernels, Localized Refinement are also possible. It is worth to note that, the domain of dependence of MQ is global. This means that they are non-compactly supported functions; they do not vanish at any finite distance from the center while Laguerre Gaussians are compactly supported in practice (though not strictly), due to the Gaussian term $e^{(\varepsilon r)^2}(2-(\varepsilon r)^2)$, which causes rapid decay as ε increases [15].

2.3. Effects of Increasing Collocation Points.

Improved accuracy and convergence as the solution is better approximated. Increasing the number of collocation points leads to a finer discretization of the problem domain, reducing the fill distance h. This improves the accuracy and the rate convergence of the method as the collocation points more densely cover the domain, leading to a better approximation of the solution. Though, the system becomes open to ill-conditioning and numerical instability if the system becomes poorly scaled. On othe considerations such as what will be the corresponding impact if the kernel is weakly singular or unbounded, (see [27], [28] and [29]).

3. Numerical Results

Problem 1: Consider the linear Volterra integro-differential equation:

$$u^{(iv)}(x) = x(1+e^x) + 3e^x + u(x) - \int_0^x u(y)\,dy \quad (9)$$

subject to the boundary conditions:

$u(0) = 1, u'(0) = 1, u(1) = 1 + e$, and $u'(1) = 2e$ on the interval [0,1]. The exact solution is $u(x) = 1 + xe^x$ and

Problem 2: Consider the linear Volterra integro-differential equation taken from [26]:

$$u^{(\prime\prime)}(x) = -x - \frac{x^3}{6} + \int_0^x (x-y)u(y)\,dy \quad (10)$$

subject to the boundary conditions:

$u(0) = 0, u'(0) = 2, u(1) = 1 + \sin 1$, and $u'(1) = 1 + \cos 1$ on the interval [0,1]. The exact solution is $u(x) = x + \sin x$ [26]. Here, $k = 4$, $\alpha = \beta = 2$, $a = 0$, that is $\alpha = \left\lfloor \frac{4}{2} \right\rfloor = 2$ and $\beta = \left\lfloor \frac{4}{2} \right\rfloor = 2$ on the interval [0, 1]. These problems were solved for $m = 17$ and 23 respectively. Thus, the collocation points will be $x_2, \ldots x_{10}$, $x_2, \ldots, x_{16}$ and $x_2, \ldots, x_{22}$ respectively. We use linear Laguerre-Gaussians, and generalized Multiquadrics kernel.

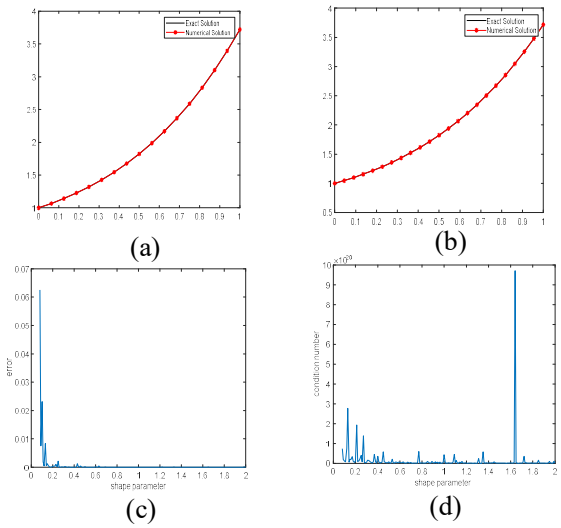

Figure 1. Numerical and Exact Solution for problem 1, and Behaviour of Shape Parameter with respect to (c) Error and (d) Condition Number, using Linear Laguerre-Gaussian Kernel.

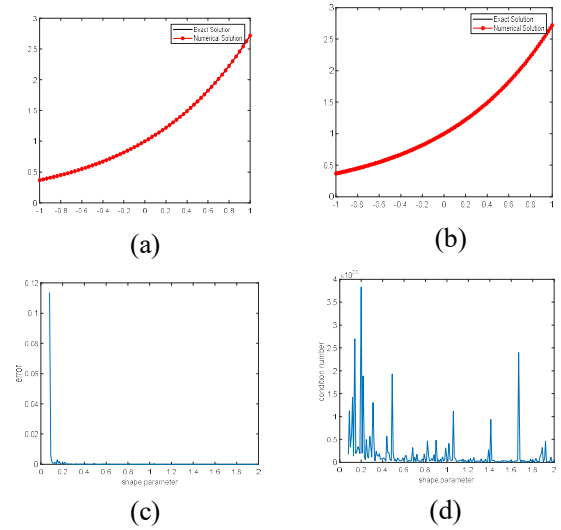

Figure 2. Numerical and Exact Solution for problem 2, and Behaviour of Shape Parameter with respect to (c) Error and (d) Condition Number, using Generalized Multi-Quadrics Kernel.

Table 1. Numerical Solution and Absolute Error for the Problem 1, using Linear Laguerre-Gaussians and Generalized Multi-quadrics ($\boldsymbol{m = 17}$).

Points	x_i	Exact Solution	Approximate Solution (LLG)	Absolute Error (LLG)	Approximate Solution (GMQ)	Absolute Error (GMQ)
1	-1	0.367879	0.367879	8.968316×10^{-7}	0.367879	2.212018×10^{-6}
2	-7/8	0.416862	0.416862	2.7447000×10^{-8}	0.416862	5.183214×10^{-8}
3	-6/8	0.472367	0.472367	3.655369×10^{-7}	0.472367	8.563639×10^{-8}
4	-5/8	0.535261	0.535261	4.737887×10^{-8}	0.535261	8.148962×10^{-9}
5	-4/8	0.606531	0.606531	6.810450×10^{-8}	0.606531	1.080118×10^{-7}
6	-3/8	0.687289	0.687289	1.156636×10^{-7}	0.687289	4.004509×10^{-8}
7	-2/8	0.778801	0.778801	6.596934×10^{-8}	0.778801	7.288332×10^{-8}
8	-1/8	0.882497	0.882497	1.667250×10^{-7}	0.882497	5.581526×10^{-8}
9	0	1.000000	1.000000	1.096705×10^{-7}	1.000000	1.101389×10^{-7}
10	1/8	1.133148	1.133148	1.227157×10^{-7}	1.133148	1.491695×10^{-8}
11	2/8	1.284025	1.284025	1.402587×10^{-7}	1.284025	8.583456×10^{-8}
12	3/8	1.454991	1.454991	9.548872×10^{-8}	1.454991	1.270210×10^{-7}
13	4/8	1.648721	1.648721	1.933080×10^{-7}	1.648721	7.106465×10^{-7}
14	5/8	1.868246	1.868246	1.624123×10^{-7}	1.868246	1.817695×10^{-8}
15	6/8	2.117000	2.117000	2.188568×10^{-7}	2.117000	4.305514×10^{-8}
16	7/8	2.398875	2.398875	1.448256×10^{-7}	2.398875	1.317918×10^{-7}
17	1	2.718279	2.718282	2.285417×10^{-6}	2.718282	4.434408×10^{-6}

Table 1. Numerical Solution and Absolute Error for the Problem 2, using Linear Laguerre-Gaussians and Generalized Multi-quadrics ($\boldsymbol{m = 23}$).

Points	x_i	Exact Solution	Approximate Solution (LLG)	Absolute Error (LLG)	Approximate Solution (GMQ)	Absolute Error (GMQ)
1	-1	0.367879	0.367879	2.164776×10^{-7}	0.367879	2.444864×10^{-7}
2	-10/11	0.402890	0.402890	5.226292×10^{-8}	0.402890	4.787868×10^{-7}
3	-9/11	0.441233	0.441233	2.639722×10^{-7}	0.441233	3.658647×10^{-7}
4	-8/11	0.483225	0.483225	1.839251×10^{-8}	0.483225	3.312609×10^{-7}
5	-7/11	0.529213	0.529213	1.995590×10^{-7}	0.529213	4.529079×10^{-7}
6	-6/11	0.579578	0.579578	1.344108×10^{-7}	0.579578	2.164504×10^{-7}
7	-5/11	0.634736	0.634736	2.766444×10^{-7}	0.634736	3.771690×10^{-7}
8	-4/11	0.695144	0.695144	2.982010×10^{-8}	0.695144	1.214471×10^{-7}
9	-3/11	0.761300	0.761300	2.000703×10^{-7}	0.761300	1.009426×10^{-7}
10	-2/11	0.833753	0.833753	1.750247×10^{-8}	0.833753	1.633778×10^{-7}
11	-1/11	0.913101	0.913101	6.388558×10^{-8}	0.913101	7.196382×10^{-8}
12	0	1.000000	1.000000	3.329478×10^{-8}	1.000000	3.230525×10^{-8}
13	1/11	1.095169	1.095169	7.330447×10^{-8}	1.095169	3.552203×10^{-7}
14	2/11	1.199396	1.199396	2.137575×10^{-8}	1.199396	1.905272×10^{-7}
15	3/11	1.313542	1.313542	1.016782×10^{-8}	1.313542	3.051642×10^{-7}
16	4/11	1.438552	1.438551	9.354295×10^{-8}	1.438551	6.030863×10^{-7}
17	5/11	1.575458	1.575457	9.003274×10^{-8}	1.575457	5.254260×10^{-7}
18	6/11	1.725393	1.725392	5.534307×10^{-8}	1.725392	2.355539×10^{-7}
19	7/11	1.889598	1.889597	9.345523×10^{-8}	1.889597	6.415385×10^{-7}
20	8/11	2.069429	2.069429	2.162231×10^{-8}	2.069429	2.921715×10^{-7}
21	9/11	2.266376	2.266375	4.565158×10^{-8}	2.266375	7.227230×10^{-7}
22	10/11	2.482066	2.482065	2.223534×10^{-7}	2.482065	5.529729×10^{-7}
23	1	2.718282	2.718282	4.995921×10^{-9}	2.718282	4.538934×10^{-7}

4. Discussion

The results for the radial kernel collocation solution of problem 1 and problem 2 using linear Laguerre Gaussian and the generalized multi-quadrics for $m = 17$ and $m = 23$ are provided in Table 1 and Table 2 respectively. Other numerical results are display in form of graphs shown in Figure 1 and Figure 2. In Table 1 with $m = 17$, we observed that the differences in absolute error for the two radial kernels are not much. However, the approximations using linear Laguerre Gaussian are generally more accurate than the approximation by generalized multiquadrics except at the origin where the generalized multiquadrics performs better. On a more general level, the approximation for $m = 17$ was not very accurate compared to higher values. It was also observed from graph of the behaviour of shape parameter against error and that of the shape parameter against the condition number of the system matrix in Figures 1 and 2 that, the error was more pronounced around the origin with small values of the shape parameter while the system matrix was badly conditioned at many points on the domain using Gaussian kernel for $m = 17$. While for the linear Laguerre-Gaussian kernel, the error experience a shoot for values of ε between 0 and 0.4 and the system matrix was also badly conditioned but yet offer a good approximation. For linear Laguerre Gaussian the value of the shape parameter $\varepsilon = 0.6$ to 2 give the most accurate results while for the generalized multiquadrics $\varepsilon = 0.25$ to 2 were the most suitable.

5. Conclusion

In order to solve the Volterra integro-differential equations for a better conditioned system and accuracy, the radial kernels collocation method was developed in this paper employing two distinct positive radial kernels. Two examples based on the suggested algorithm were used to demonstrate the efficacy of the approach. Additionally, the method's convergence was provided in relation to the shape parameter's behavior with respect to error and condition number. The findings demonstrate that, the proposed method is realistically dependable and consistent when compared to the other approaches such as the traditional methods. As a result, employing the radial kernels method to solve integro-differential equation problems improves accuracy rates, which is encouraged.

Conflicts of Interest Statement

We declare that there are no conflicts of interest regarding the publication of this research. No sources of funding, affiliations, and any relevant financial or non-financial relationships have influenced this research or its outcomes.

Data Availability Statement

All sources of data and information used for this research have been duely acknowlewdged and a list of reference provided.

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