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Dynamics and Bifurcation for One Non-linear System

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Abstract

In this paper, we observed the ordinary differential equation (ODE) system and determined the equilibrium points. To characterize them, we used the existing theory developed to visualize the behavior of the system. We describe the bifurcation that appears, which is characteristic of higher-dimensional systems, that is when a fixed point loses its stability without colliding with other points. Although it is difficult to determine the whole series of bifurcations that lead to chaos, we can say that it is a common opinion that it is precisely the Hopf bifurcation that leads to chaos when it comes to situations that occur in applications. Here, subcritical and supercritical bifurcation occurs, and we can say that subcritical bifurcation represents a much more dramatic situation and is potentially more dangerous than supercritical bifurcation, technically speaking. Namely, bifurcations or trajectories jump to a distant attractor, which can be a fixed point, limit cycle, infinity, or in spaces with three or more dimensions, a foreign attractor.

Keywords: *equilibrium, bifurcation, stability, dynamics.*

1. Introduction

When analyzing a nonlinear system of ordinary differential equations, it is very important to examine the conditions under which bifurcations occur. We deal with the analysis of the Hopf bifurcation. It arises in the case when the limit cycle or periodic solution, which surrounds the equilibrium, appears or disappears with a change in the value of the parameter. Often, to examine the Hopf bifurcation, the starting system is converted to polar coordinates. It is not always easy, but it is important to establish whether it is a supercritical or a subcritical Hopf bifurcation. If a stable limit cycle surrounds an unstable equilibrium point, then the bifurcation is called a supercritical Hopf bifurcation, and conversely, if an unstable limit cycle surrounds a stable equilibrium point, then the bifurcation is called a subcritical Hopf bifurcation. It is impossible in the general case to determine whether a supercritical or subcritical Hopf bifurcation occurs by linearization alone.

In this paper, we investigate the behavior of a system of differential equations that has the form

$$\begin{cases} \dot{x} = -ax + y + y^2, \\ \dot{y} = -\mu x + (\mu + 4)y - xy + (\mu - 2)y^2. \end{cases} \tag{1}$$

Where a is a positive real parameter and μ a real parameter. We want to examine the dynamics of the mentioned system and point out the important phenomena that will appear. We encounter the study of such systems in the dynamics of nerve cells, aeroelastic shaking, aerodynamic vibrations, and the occurrence of instability in fluid flow. A special motivation for studying this kind of system is that Hopf bifurcation appears in it, although it is only of the second order, usually, this phenomenon occurs in higher-order systems.

The investigation of this system is interesting from the point of view of the appearance of the Hopf bifurcation. The appearance of the Hopf bifurcation represents the basic beginning of the quasi-periodic path to chaos, when the spiral node becomes unstable, and passes into the limit cycle through the Hopf bifurcation, after which chaos appears. The study of bifurcation can be found in papers [1]-[4], in addition to bifurcations associated with continuous systems, this phenomenon also occurs in

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discrete systems, where the common name is Neimark–Sacker bifurcation, an example of such bifurcations is in papers [5]-[7].

The application of the theory of nonlinear dynamics in activity modeling is of great importance because it provides a clear formalism and provides the possibility for recognition and classification of universal characteristics of the observed systems. In this paper, we use the theory developed in paper [8], [9], [10], and [11]. The study and significance of the ordinary differential equation (ODE) system were in the works [12]-[17] and the discrete system was in the works [5]-[7].

2. Analysis of system dynamics

To characterize the observed system (1), let's determine the equilibrium points. We get them by solving equations

$$\begin{aligned} -ax + y + y^2 &= 0, \\ -\mu x + (\mu+4)y - xy + (\mu-2)y^2 &= 0. \end{aligned} \tag{2}$$

Theorem 1. The solutions of the system (2) are points $O(0,0)$, $A(\frac{1}{2a}\Big(a^2(-2+\mu)^2 + \mu\Big(-1+\mu+\sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)}\Big) + a\Big(2\Big(5+\sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)}\Big) - \mu\Big(-5+2\mu+\sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)}\Big)\Big)\Big),$ $\frac{1}{2}(-1+a(-2+\mu)-\mu-\sqrt{(1-a(-2+\mu)+\mu)^2+4(-\mu+a(4+\mu))})),$ $B(\frac{1}{2a}\Big(a^2(-2+\mu)^2+(-1+\mu)\mu+a(10+(5-2\mu)\mu) - 2a\sqrt{(1-a(-2+\mu)+\mu)^2+4\big(-\mu+a(4+\mu)\big)} - \mu\sqrt{(1-a(-2+\mu)+\mu)^2+4\big(-\mu+a(4+\mu)\big)} + a\mu\sqrt{(1-a(-2+\mu)+\mu)^2+4\big(-\mu+a(4+\mu)\big)}\Big), \frac{1}{2}(-1+a(-2+\mu)-\mu+\sqrt{(1-a(-2+\mu)+\mu)^2+4(-\mu+a(4+\mu))})).$

These points are equilibrium points for system (1).

Proof: From the second equation of system (2) we can find $x=\frac{4y-2y^2+y\mu+y^2\mu}{y+\mu}$. Inserting into the first equation in (2) we find that $y\big((1+y)(y+\mu) - a(4+y(-2+\mu)+\mu)\big) = 0$. From the above, we obtain the statement of the theorem.

To apply the theory developed for the study of such systems, let's write the observed system in matrix form

$$\dot{\boldsymbol{x}} = \begin{pmatrix} -ax+y+y^2 \\ -\mu x + (\mu+4)y - xy + (\mu-2)y^2 \end{pmatrix}.$$

The Jacobian matrix associated with this mapping is given in the form

$$D_f = \begin{pmatrix} -a & 1+2y \\ -y-\mu & 4-x+2y(-2+\mu)+\mu \end{pmatrix}. \tag{3}$$

Let's calculate the mapping value of (3) at the point $O(0,0)$.

$$D_f(O) = \begin{pmatrix} -a & 1 \\ -\mu & 4+\mu \end{pmatrix}.$$

The value of the trace and the determinant is $tr\big(D_f(O)\big) = 4 - a + \mu$ and $\det\big(D_f(O)\big) = -4a + \mu - a\mu$. From this, we see that it is $\det\big(D_f(O)\big) < 0$ for $a(4+\mu) > \mu$ or $a = 1$. It is similar $\det\big(D_f(O)\big) > 0$ for $a(4+\mu) < \mu$ *and* $a \neq 1$. By direct calculation, we find that $\det\big(D_f(O)\big) - \frac{1}{4}\Big(tr\big(D_f(O)\big)\Big)^2 = \frac{1}{4}(-16-8a-a^2-4\mu-2a\mu-\mu^2) < 0$.

From all of the above, we can conclude that the following theorem is valid:

Theorem 2. The character of point $O(0,0)$ is:

1. Saddle point for $a(4+\mu) > \mu$ or $a = 1$.
2. Nodal source for $a < 1$ *and* $a(4+\mu) < \mu$.
3. Nodal sink for $a > 1$ *and* $a(4+\mu) < \mu$.
4. Center for $\mu = 4 - a$.
5. Comb for $\mu = \frac{4a}{1-a}$ *and* $a \neq 1$.

Let's examine the behavior of point A. In this case we find that it is

$\det\big(D_f(A)\big) = \frac{1}{2}(1 + a^2(-2+\mu)^2 + \sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)} + \mu(-2+\mu+\sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)}) +$

$$a\left(2\left(10+\sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)}\right)-\mu\left(-6+2\mu+\sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)}\right)\right)\right)$$

and $tr\left(D_f(A)\right)=\frac{1}{2a}\Big(\mu+a^2(2+(-4+\mu)\mu)-a(-2+\mu)\left(1+\sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)}\right)-\mu\left(\mu+\sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)}\right)\Big).$

We can see that the obtained expressions are quite large and not simple for some calculations. However, we can notice that it is always $\det\left(D_f(A)\right)>0$. For $\mu<0$, we have that is always $tr\left(D_f(A)\right)>0$. Based on this consideration, we conclude that the theorem is valid.

Theorem 3. The equilibrium point A is the source for $\mu<0$, and $\mu>0$ and $tr\left(D_f(A)\right)>0$. For $\mu>0$ and $tr\left(D_f(A)\right)<0$, this point is the sink.

Let's complete our research for point B as well. It's worth it

$$\det\left(D_f(B)\right)=\frac{1}{2}\Big(1+a^2(-2+\mu)^2-\sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)}+\mu\left(-2+\mu-\sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)}\right)+a\left(20-2\sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)}+\mu\left(6-2\mu+\sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)}\right)\right)\Big)$$

and $tr\left(D_f(B)\right)=\frac{1}{2a}\Big(a^2(2+(-4+\mu)\mu)+a(-2+\mu)\left(-1+\sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)}\right)+\mu\left(1-\mu+\sqrt{a^2(-2+\mu)^2+(-1+\mu)^2-2a(-5+\mu)(2+\mu)}\right)\Big).$

By direct checking, we can see that it will be valid $\det\left(D_f(B)\right)<0$ for $a(4+\mu)<\mu$ and $a\neq 1$, $\det\left(D_f(B)\right)>0$ for $a(4+\mu)>\mu$ and $a=1$. Using the given facts, we can say that the theorem holds.

Theorem 4. For $a(4+\mu)<\mu$ and $a\neq 1$ equilibrium point B is the saddle point. For $a(4+\mu)>\mu$ and $a=1$, and in addition to that $tr\left(D_f(B)\right)>0$ ($tr\left(D_f(B)\right)<0$) this pint is a source (sink).

To better understand and describe the behavior of the observed system, we perform a simulation for some parameter values. From the graphic representations in Figure 1 and Figure 2, we can see that the essential values of the parameters of our system are $a=1$ and $\mu=1$.

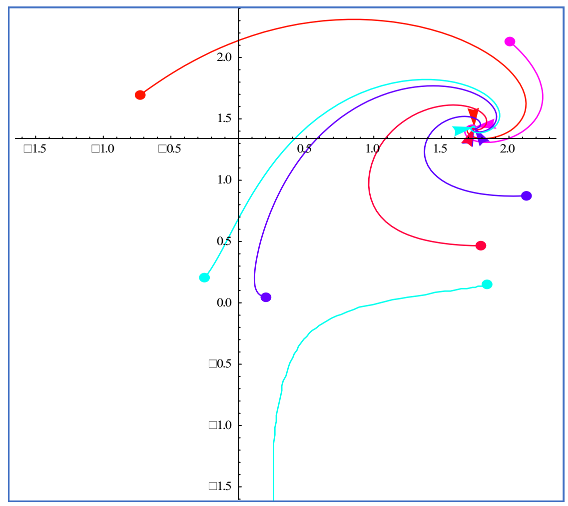

Figure 1. The behavior of the solution for $\boldsymbol{a}=\mathbf{2}$ $\mu=\mathbf{0.5}$.

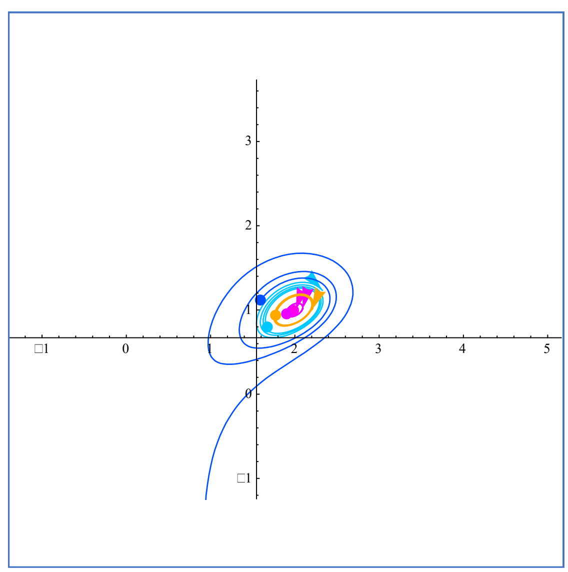

Figure 2. The behavior of the solution for $a=1$ $\mu=1$.

We notice that orbits appear for these parameter values, i.e. that depending on the starting point, some solutions go in and some out. This means that orbits and some bifurcations can appear here. Let's enter the mentioned values into the equilibrium points. We get now the following values $A_1(12,-2), \mathrm{B}_1(2,1)$. We pay special attention to point B_1 because, according to the simulations, there is a possible occurrence of bifurcations, which we want to investigate.

If we determine the eigenvalues of $D_f(B_1)$ we have the following equation:

$$\lambda_{1,2} = \frac{(-3+3\mu \pm \sqrt{-11-18\mu+9\mu^2})}{2}.$$

We can state that the eigenvalues are conjugately complex if $\mu \in \left(\frac{1}{3}(3-2\sqrt{5}), \frac{1}{3}(3+2\sqrt{5})\right)$. A well-known algorithm for examining the Hof bifurcation is given:

1. $\alpha(\mu_0) = 0, \beta(\mu_0) = \omega \neq 0$,

$sgn(\omega) = sgn\frac{\partial\, g_\mu}{\partial\, x}(x_0, y_0)$ for $\mu = \mu_0$,

$\frac{\partial\, \alpha(\mu)}{\partial\, \mu} = d \neq 0$, for $\mu = \mu_0$,

$$f = \frac{1}{16}\left[\frac{\partial^3 f_\mu}{\partial x^3}(x_0,y_0) + \frac{\partial^3 f_\mu}{\partial x \partial y^2}(x_0,y_0) + \frac{\partial^3 g_\mu}{\partial y^3}(x_0,y_0)\right] + \frac{1}{16}\left[\frac{\partial^2 f_\mu}{\partial x \partial y}(x_0,y_0)\left(\frac{\partial^2 f_\mu}{\partial x^2}(x_0,y_0) + \frac{\partial^2 f_\mu}{\partial y^2}(x_0,y_0)\right) - \frac{\partial^2 g_\mu}{\partial x \partial y}(x_0,y_0)\left(\frac{\partial^2 g_\mu}{\partial x^2}(x_0,y_0) + \frac{\partial^2 g_\mu}{\partial y^2}(x_0,y_0)\right) - \frac{\partial^2 f_\mu}{\partial x^2}(x_0,y_0)\frac{\partial^2 g_\mu}{\partial x^2}(x_0,y_0) + \frac{\partial^2 f_\mu}{\partial y^2}(x_0,y_0)\frac{\partial^2 g_\mu}{\partial y^2}(x_0,y_0)\right]$$

for $\mu = \mu_0$,

2. $f \neq 0$.

We used labels f_μ and g_μ for system equations, i.e. the observed system is of the form:

$$\begin{cases} \dot{x} = f_\mu(x,y), \\ \dot{y} = g_\mu(x,y). \end{cases}$$

When we apply the mentioned algorithm, in our case we have the following equation:

$$\alpha(\mu) = \frac{3\mu - 3}{2}, \beta(\mu) = \frac{\sqrt{11+18\mu-9\mu^2}}{2}.$$

Calculating directly we find $\alpha(1) = 0,\ \beta(1) = \sqrt{5}$.

$\frac{\partial\, g_\mu}{\partial\, x}(2,1) = -2$, well it is finally $\omega = -\sqrt{5}$,

$d = \frac{3}{2} \neq 0$ and $f = \frac{3(2-\mu)}{8\sqrt{5}}$, for $\mu = 1$, $f = \frac{3}{8\sqrt{5}} \neq 0$.

From the coefficients obtained in this way, we conclude that the conditions for the existence of Hopf bifurcation are fulfilled. According to [14] there is a bifurcation of the unique curve of periodic solutions from a fixed point into the region $\mu < 2$, because $fd < 0$ and into the region $\mu > 2$, because $fd > 0$. Periodic orbits are asymptotically stable for $f < 0$, and unstable for $f > 0$. We can conclude that we have supercritical bifurcation and subcritical bifurcation. We can formulate all this in the following theorem.

Theorem 5. If $\mu < 2$ then

a) a limit cycle is asymptotically stable,
b) Hopf bifurcation is supercritical.

If $\mu > 2$ then

c) a limit cycle is unstable,
d) Hopf bifurcation is subcritical.

Figure 3 presents the occurrence of bifurcation and Figure 4 presents the bifurcation diagram.

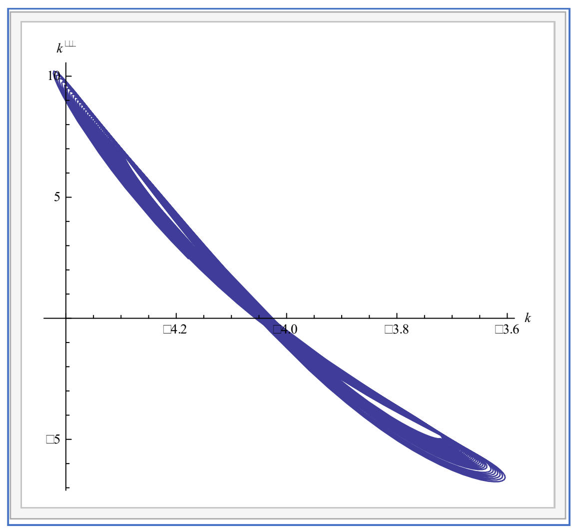

Figure 3. Occurrence of bifurcation for μ = **1**.

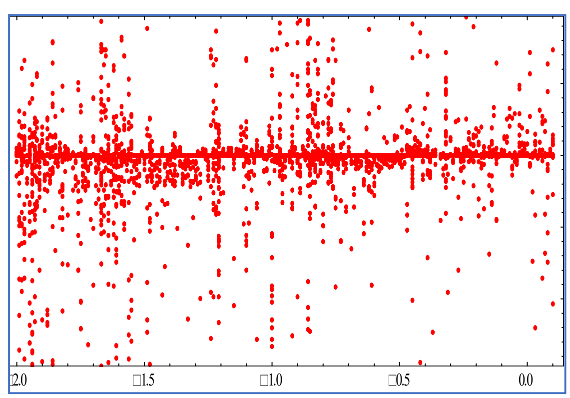

Figure 4. Bifurcation diagram.

3. Conclusion

In this paper, we see that Hopf bifurcation occurs even with a system of quadratic equations with two parameters, where one of the parameters affects its occurrence. By varying this parameter, the bifurcation changes from supercritical to subcritical. Studying the occurrence of bifurcation is interesting but also complex, depending on the number of parameters that appear. In our research, we described the local dynamics of the considered system, using simulations, and observed that bifurcation is possible, which we then proved and presented in Figure 3 and Figure 4. From the above, we can see that our system has complex and interesting dynamics.

In some future research, it would be desirable to examine whether equilibrium points with negative coordinates are also applicable in some areas. In addition, these results provide theoretical foundations on which research related to some diseases can be based.

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